gp = group

fct = function

alg'm = algorithm

\mathcal $\mathcal{D}$ $\mathcal{F}$

\mathfrak $\mathfrak{S}$ $\mathfrak{g}$

(Hodge ⟿ p-adic Hodge ⟿) Hodge-Arakelov theory
③ motivation of Θ-link
application
(absolute) anabelian geometry
① inter-universal geometry
(logically no need)
④ application
② philosophical view point
(complex, p-adic) Teichmüller theory
Diophantine inequality
① ~ ④
Mochizuki's discoveries

① anab ⇝ int-univ.

F : field

Galois theory : $G_F \supset G_{K_1} \,\|\, \supset G_{K_2} \Rightarrow K_1 = K_2$

anabelian geom

Neukirch-Uchida $K_1, K_2 / \mathbb{Q}$ fin.

$G_{K_1} \cong G_{K_2} \Rightarrow K_1 \cong K_2$

as top. gp   as fields

X_1, X_2 : hyperb. curves / field F

X_1, X_2 : hyperb. curves / field F

Under some conditions

outer isoms — arith. fund. gp

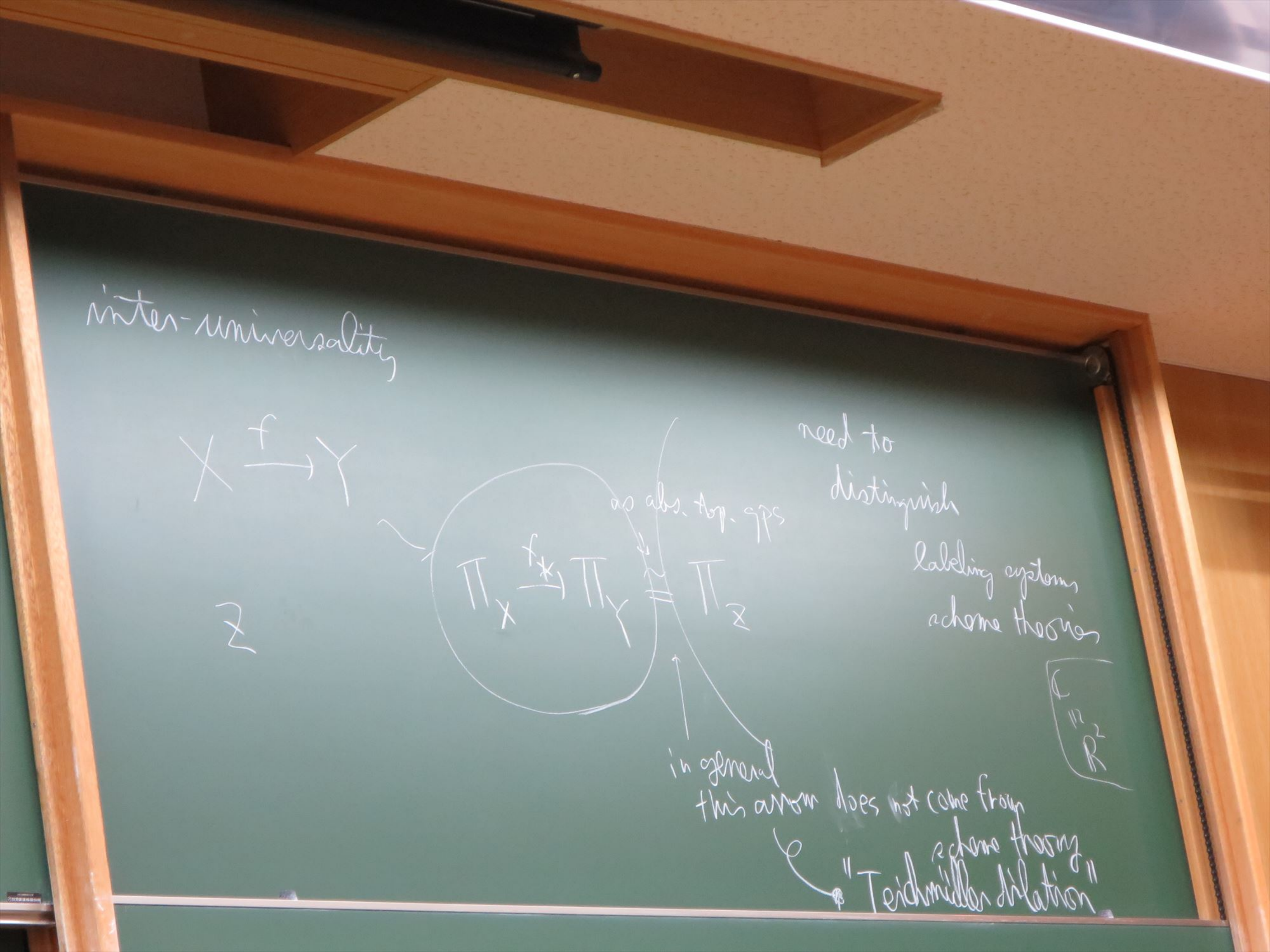
$$\mathrm{Isom}_F(X_1, X_2) \xrightarrow{\sim} \mathrm{Isom}^{\mathrm{out}}_{G_F}(\Pi_{X_1}, \Pi_{X_2})$$

(Hom ...)

Q. When do we treat Π_X or G_F as abstract top. gps?

A. In IUTch!!

(possibly first time)


inter-universality
X →f Y
Z
Π_X →f_* Π_Y ≅ Π_Z
as abs. top. gps
need to distinguish
labeling systems
scheme theories
ℂ ≅ ℝ²
in general
this arrow does not come from
scheme theory
"Teichmüller dilation"

② anab $\overset{\text{Teich}}{\curvearrowright}$ inter-univ.

$F_1, F_2 / \mathbb{Q}_p$ fin.

$$\mathrm{Isom}(\overline{F}_1/F_1, \overline{F}_2/F_2)$$

$$\subsetneq \mathrm{Isom}(\mathrm{Gal}(\overline{F}_2/F_2), \mathrm{Gal}(\overline{F}_1/F_1))$$

$$\mathrm{Aut}(F) \subsetneq \mathrm{Out}(G_F)$$

cf. [NSW] Chap. VII §5 P. 420–423

∃ outer autom's which do not come from isom's of top'l fields

$\mathrm{codim}(G_F) = 2$

$$1 \longrightarrow I_F \longrightarrow G_F \longrightarrow \hat{\mathbb{Z}}\,\mathrm{Frob}_F \longrightarrow 1$$

I_F: ∃ autom's which do not come from field theory — inter-universally non-rigid

Frob_F: can recover gp th'cally — inter-universally rigid

LCFT

$0 \to O_F^\times \to F^\times \xrightarrow{val} \mathbb{Z} \to 0$

$O_F^\times$ ↑ non-rigid

$\mathbb{Z}$ ↑ rigid

F: ⊞ ⊠

$G_F \curvearrowright F^\times$

can recover only ⊠

cannot recover ⊞

cf. $\mathrm{Isom}(F_1, F_2) \cong \mathrm{Out}_{\text{ram. fil.}}(G_{F_2}, G_{F_1})$

[φGC]

Teichmüller dilation

$\mathrm{codim}(G_F) = 2$

rob$_F$ → 1

ran rècomer gp th'cally

ry

inter-universally rigid

$\mathbb{C}^\times = S^1 \times \mathbb{R}_{>0}$

under autom's of $\mathbb{C}^\times$

rigid non-rigid

MLF : $\exists$ Neukirch-Uchida type thm

sad thing $\rightsquigarrow$ joy

abs. anab. geom. $\qquad$ $G_F \overset{\text{alg'm}}{\rightsquigarrow}$ mult. gp $F^\times$

Belyi cuspidalization

X : strictly Belyi type $/F$ $\qquad$ $\Pi_X \rightsquigarrow$ top. field F

$F/\mathbb{Q}_p$ fin.

G_F , $G_F \curvearrowright \mathcal{O}_{\overline{F}}^{\triangleright} := \mathcal{O}_{\overline{F}} \setminus \{0\}$ "mono-analytic" ($\rightsquigarrow$ real analytic)

Π_X , $\Pi_X \curvearrowright \mathcal{O}_{\overline{F}}^{\triangleright}$ "arithmetically holomorphic" ($\rightsquigarrow$ holomorphic)

regard Π_X as "tangent vector" of F

$\hookrightarrow$ rigidify the non-rigid dimension

③ Hodge-Arakelov ⇝ inter-univ.

p-adic Hodge

Hodge

Thm of dR / $\mathbb{C}$

$$H_1(\mathbb{C}^\times, \mathbb{Z}) \otimes H^1_{dR}(\mathbb{C}^\times) \longrightarrow \mathbb{C}$$

$$\circlearrowleft \otimes \frac{dT}{T} \longmapsto \int_{\circlearrowleft} \frac{dT}{T} = 2\pi i$$

perfect pairing

p-adic Hodge / $\mathbb{Q}_p$

p-adic Hodge / $\mathbb{Q}_p$

$$T_p \mathbb{G}_m \quad \otimes H^1_{dR}(\mathbb{G}_m/\mathbb{Q}_p) \longrightarrow B_{crys}$$

p-adic Tate module ||

$$\varprojlim_n \mathbb{G}_m[p^n](\overline{\mathbb{Q}}_p) \qquad \underline{\varepsilon} \otimes \frac{dT}{T} \longmapsto \text{"}\int_{\underline{\varepsilon}} \frac{dT}{T}\text{"} = \log[\underline{\varepsilon}] = t$$

$(2\pi i)$

$$\underline{\varepsilon} = (\varepsilon_n)_n$$

$$\varepsilon_1 = 1,\ \varepsilon_2 \neq 1,\ \varepsilon_{n+1}^p = \varepsilon_n$$

"analytic path"

$E / \mathbb{Z}_p$ ell. curve

$$T_p E \otimes H^1_{dR}(E/\mathbb{Q}_p) \longrightarrow B_{crys}$$

$\parallel$ $\mathrm{coLie}\, E^{\dagger}$

$$\underline{P} \otimes \omega \longmapsto \text{“} \int_{\underline{P}} \omega \text{”}$$

$$\underline{P} = (P_n)_n \quad P_1 = 0,\ p P_{n+1} = P_n \qquad \text{“} \log_\omega([\underline{P}]) \text{”}$$

“analytic path”

$$\log_\omega : \hat{E}_{/\mathbb{Q}_p} \longrightarrow \hat{\mathbb{G}}_{a,\mathbb{Q}_p}$$

$$(\log_\omega)^* dT = \omega$$

univ. ext'n

$$0 \to \mathrm{coLie}\, E^\vee \to E^\dagger \to E \to 0$$

$/\mathbb{C}$ Hodge fil $H_1(E,\mathbb{Z}) = H_1(E,\mathbb{Z})$

$$0 \to \mathrm{coLie}\, E^\vee \to \mathrm{Lie}\, E^\dagger \to \mathrm{Lie}\, E \to 0$$

$\parallel$ $(H^1_{dR})^*$ $\downarrow$

$$0 \to \mathrm{coLie}\, E^\vee \to E^\dagger \to E \to 0$$

Hodge-Arakel

Hodge-Arakelov

F : NF , E/F ell. curve , $l>2$ prime

$E(F)[2] \ni P \neq 0$, $\mathcal{L} := \mathcal{O}(l[P])$ line bdl over E $\dim_F = l^2$

Roughly

dR side — étale side

$$\Gamma(E^\dagger, \mathcal{L}|_{E^\dagger})^{\deg < l} \xrightarrow{\sim} \mathcal{L}|_{E^\dagger[l]} = \mathcal{L}|_{E[l]} = \bigoplus_{E[l]} F$$

$\dim_F = l^2$

$E^\dagger$ univ. ext., locally $\cong \mathbb{G}_{a/E} = \operatorname{Spec} \mathcal{O}_E[T]$ ← rel. deg

isom of F-vect. space & preserves specified (Arch / non-Arch) integral str.

$E/\ldots$ ell

We regard

$E[\ell]$ as an "approximation of underlying mfd of E"

(feeling $\ell \gg 0$ cf. IUT "ℓ" $\approx$ "$ht(E)$")

cf. degenerate case $\mathbb{G}_m$ case

$$F[T]^{\deg<\ell} \xrightarrow{\sim} \bigoplus_{\zeta\in\mu_\ell} F$$

$$f \longmapsto (f(\zeta))_{\zeta\in\mu_\ell}$$

Vandermonde det $\neq 0$

(consider $\mathcal{L}|_{E[\ell]}$, not $E[\ell]$ itself

"fcts on $E[\ell]$"

$\rightsquigarrow$ quantisation)

$\Gamma(E^{\dagger}, \mathcal{L}|_{E^{\dagger}})^{\deg < \ell} \xrightarrow{\sim} \mathcal{L}|_{E[\ell]}$

fil by rel. deg. $\quad Gr^{-j} \cong (\mathrm{coLie}\, E)^{\otimes(-j)}$

$\mathrm{coLie}\, E = \omega_E$

theta fil & its derivative $\longmapsfrom$ theta values

We consider int. str. Gaussian pole on the moduli of ell. curves $\mathcal{M}_{ell}$

($\omega^{\otimes 2} \cong \Omega$, $\mathcal{O}$, $2\pi i$, $\omega \mapsto \sqrt{\pi}$)

polar coord: $q^{j^2/8\ell}\, \mathcal{O}_F$ (q-parameter at bad pr.)

deg of LHS $\quad -\sum_{j=0}^{\ell-1} j\,[\omega_E] \approx -\frac{\ell^2}{2}[\omega_E] = -\frac{\ell^2}{24}[\log q]$

deg of RHS $\quad -\frac{1}{8\ell}\sum_{j=0}^{\ell-1} j^2\,[\log q] \approx -\frac{\ell^2}{24}[\log q]$

KS: $[\omega_E^{\otimes 2}] = [\Omega_{\mathcal{M}_{ell}}] = \frac{1}{6}[\log q]$

Gaussian distribution $\leftarrow$ cartesian coord.

$$\sqrt{\pi} = \int_{-\infty}^{\infty} e^{-x^2} dx$$

ω_E

cart. coord. ⤳ polar coord.

$$\omega_E^{\otimes 2} = \Omega_{\mathcal{M}_{ell}}(\log)$$

$$\int \circlearrowleft = 2\pi i$$

- anabelian geom
  ← tools
- Hodge-Arakelov "
  ← "design"
  "story"

$E[\ell] \supset M$ ← rank $= 1$, ↳ at non-arch bad $\ni \mu_\ell$

$\not\exists$ gl. mult. sub (in general)

<u>If it existed</u>

$K := F(E[\ell])$, $E' := E/M$

apply Hodge-Arakelov to E'

$q = (q_v)_{v:\text{bad}}$ q-parameter

$$\Gamma\left((E')^{\dagger}, \mathcal{L}|_{(E')^{\dagger}}\right)\Big|^{\deg < \ell} \longrightarrow \bigoplus_{-\frac{\ell-1}{2} \le j \le \frac{\ell-1}{2}} \left(q^{\frac{j^2}{2\ell}} \mathcal{O}_K\right) \otimes_{\mathcal{O}_K} K$$

Hodge fil in LHS is incompat. w/ direct sum decomp. in RHS

$\rightsquigarrow$ proj. to the j-th factor $\leftarrow$ $j \approx l \approx ht(E)$

$$Fil^0 = \underline{q}\,O_K \hookrightarrow \underline{q}^{j^2} O_K$$

non-trivial!

$\underline{q} := q^{1/2l}$

arith Kodaira-Spencer map

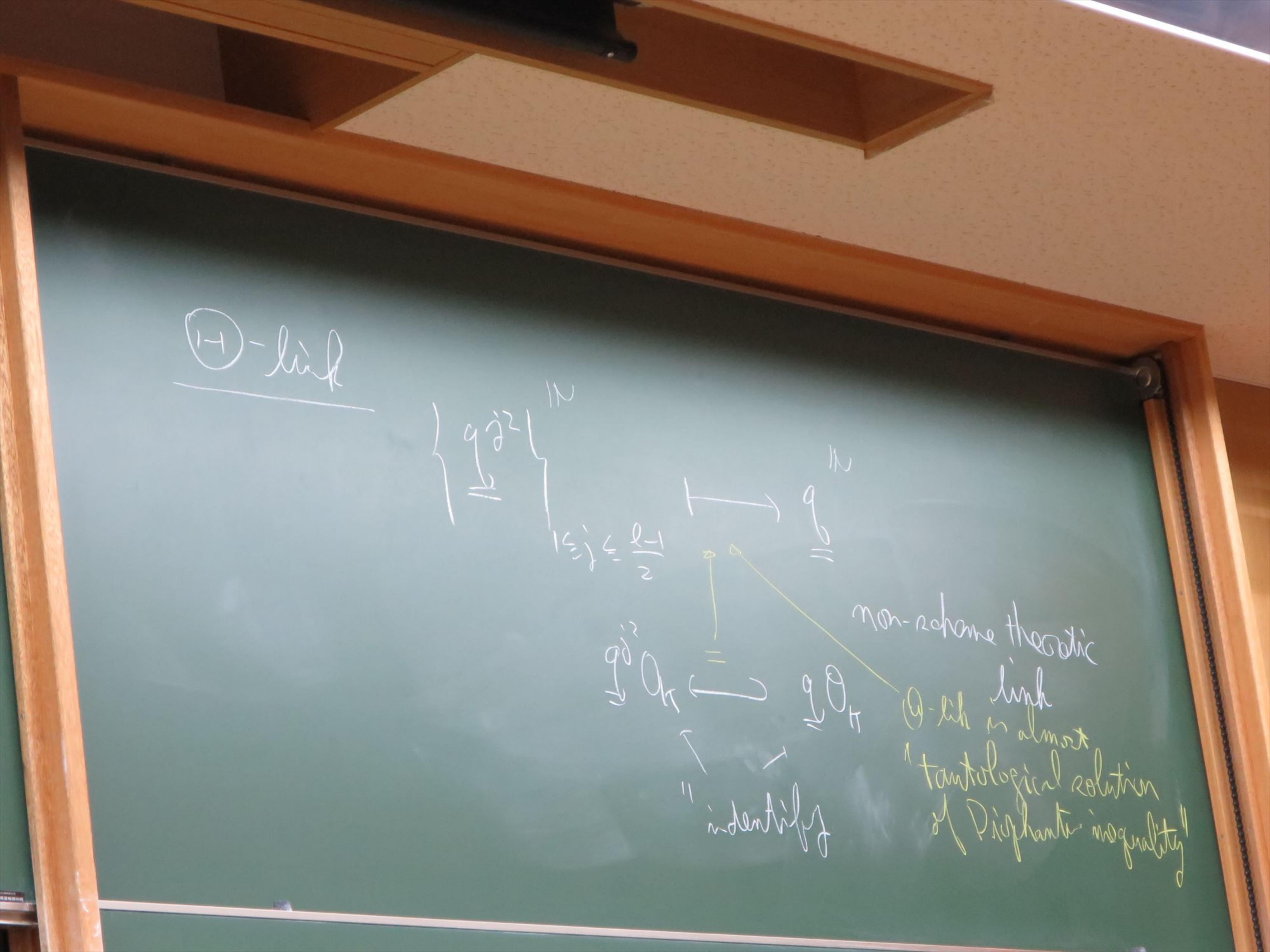
(deg $\doteqdot$ 0 arith line bdl) (deg $\ll$ 0 arith line bdl)

$\rightsquigarrow \quad 0 \lesssim -(\text{large number}) \quad (\approx -ht(E))$

$\rightsquigarrow \quad ht \lesssim 0 \quad ht$ is bounded

Want $\underline{q}\, O_K \hookrightarrow \underline{q}^{j^2} O_K \quad (*)$

- Hodge-Arakelov : use scheme theory, cannot obtain $(*)$
- IUTch : use $(*)$, abandon scheme theory


Θ-link
{q^{j^2}}^N ⟼ q^N
1 ≤ j ≤ (l-1)/2
non-scheme theoretic link
q^{j^2} Θ_π ⟷ q Θ_π
"identify"
Θ-link is almost "tautological solution of Diophantine inequality"

$$\sqrt{\pi} = \int_{-\infty}^{\infty} e^{-x^2} dx$$

cart. coord. ⤳ polar coord.

ω_E

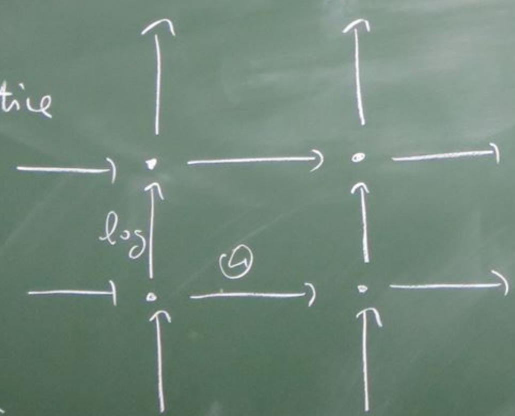
$\omega_E^{\otimes 2} = \Omega_{\overline{M}_{ell}}(\log)$

$\int \circlearrowleft = 2\pi i$

- anabelian geom ← tools
- Hodge-Arakelov ← "design" "story"

④ inter-univ ⤳ Dioph. ineq.

log-Θ-lattice



•: Hodge theater

Θ-link

$\{q^{j^2}\}_{1 \le j \le \frac{l-1}{2}}$ ← non-scheme-theoretic → q

hol fct scheme theory ← X → hol fct scheme theory

Θ-link {anab. inequality}

$2 ① = 6$

$2x = 6$

} ÷

$x = 3$

$X, Y / k = \bar{k}$ curves

$X^{top} \cong Y^{top}$ Zariski homeo

Θ

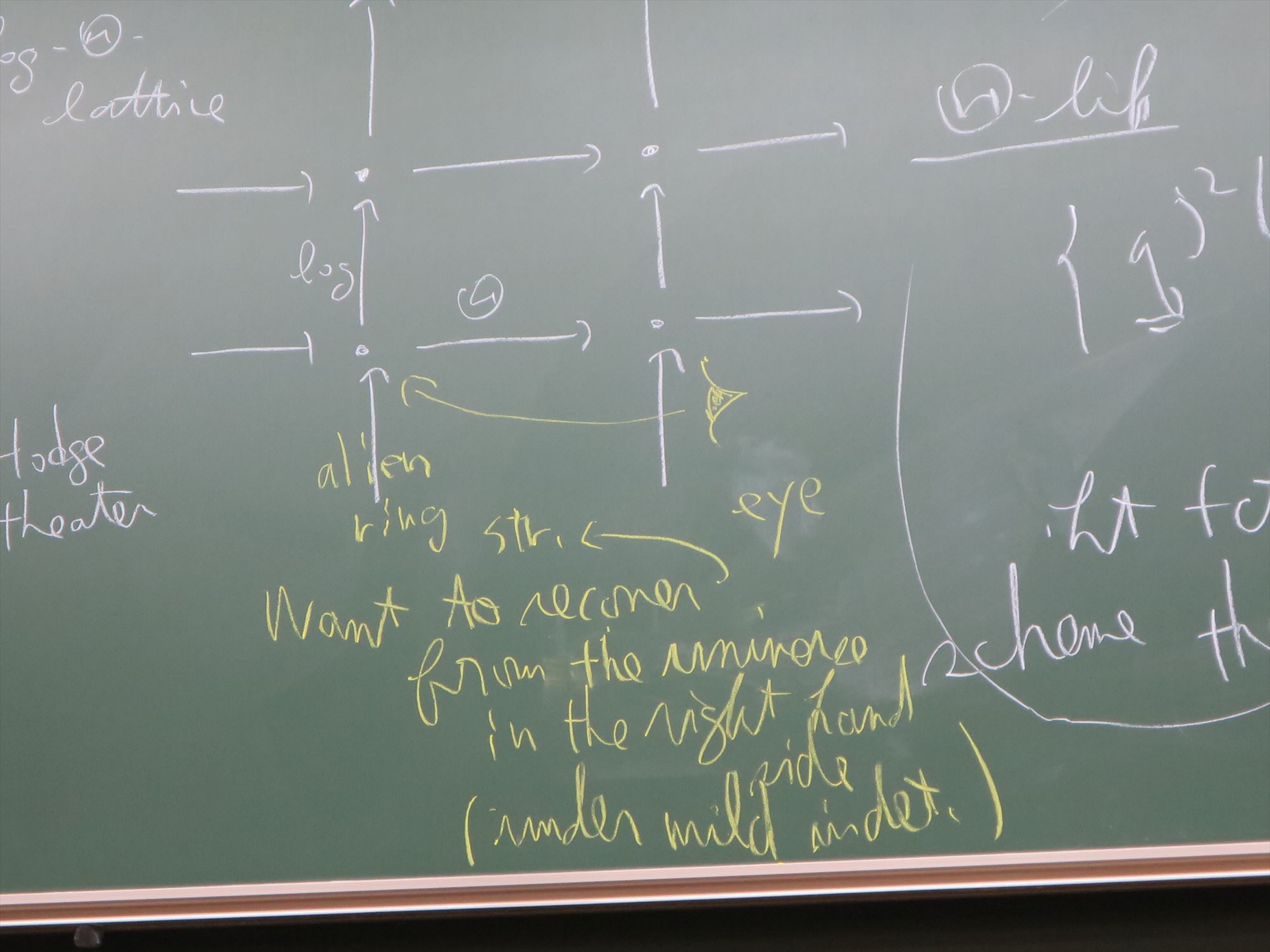
↑ : log-link

→ : Θ-link

p n h k

pTeich		IUTch
hyperb. curve / char>0	⟷	NF
indig. bdl	⟷	once punctured ell. curve / NF
Frob. in char>0	⟷	↑ log-link
lift $p^n/p^{n+1} \rightsquigarrow p^{n+1}/p^{n+2}$ in Witt ring	⟷	$\overrightarrow{\Theta\text{-link}}$

can. Frob. lift ⟷ log-Θ lattice


log-Θ-
lattice
Θ-link
log
Θ
Hodge
theater
alien
ring str.
eye
Want to recover
from the universe
in the right hand
side
(under mild indet.)
scheme

dichotomy | Frob.-like obj. : abstract top. monoids etc. — order respect → to construct links (walls)

| étale-like obj. : Π_X, G_F & obj. reconstructed from Π_X, G_F via anabelian recon. alg'm — indifferent to order → to penetrate the walls

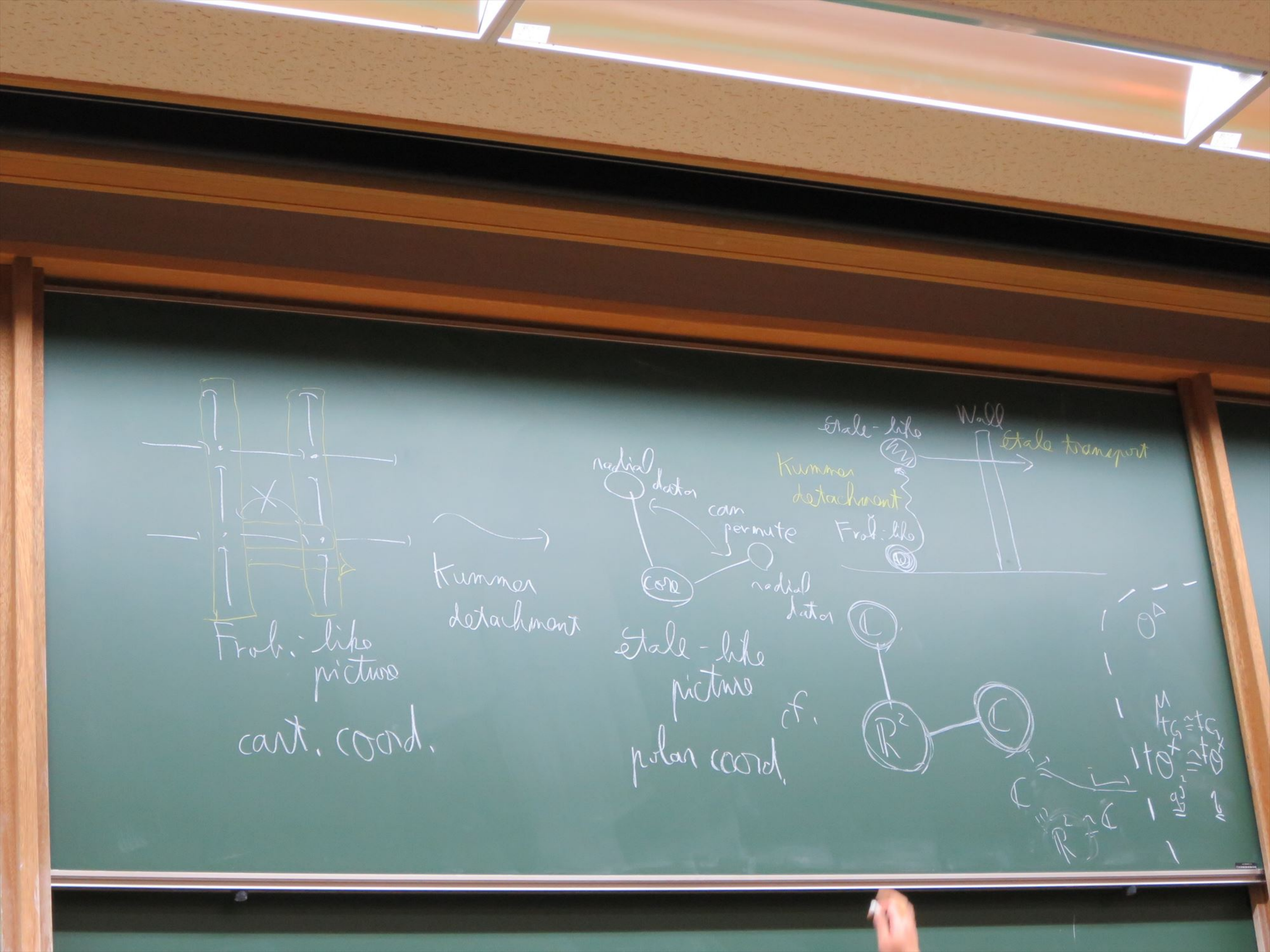
$$^{\dagger}\Pi_X \rightsquigarrow \mathcal{O}^{\triangleright}({}^{\dagger}\Pi_X) \rightsquigarrow M \;(\cong \mathcal{O}^{\triangleright}({}^{\dagger}\Pi_X))$$

$\mathcal{O}^{\triangleright}({}^{\dagger}\Pi_X)$: étale-like

M: abs. top. monoid, Frob.-like

$\mathbb{N}$

pTeich | IUTch


Frob.-like picture
cart. coord.
Kummer detachment
radial data
can permute
core
radial data
étale-like picture
cf.
polar coord.
étale-like
Kummer detachment
Frob.-like
Wall
étale transport

picture

cart. coord.

picture

cf.

polar coord.

main thm of IUTch ([IUTch III, Th 3.11])

After admitting 3 indet. (Indet $\overset{2}{\to}$), (Indet $\overset{3}{\uparrow}$), (Indet $\overset{1}{\circlearrowleft}$),

Θ-link → , log-link ↑ , ét. transp.

Kummer detach.

we can put both sides of Θ-link $\{\underline{q}^{j^2}\}_{1 \le j \le \frac{l-1}{2}} \longleftrightarrow \underline{q}$

on the same platform & we cannot distinguish them.

conseq. ⟶ can compare deg of line bdles on the both sides (under mild indet) there indet. are **mild** ones.

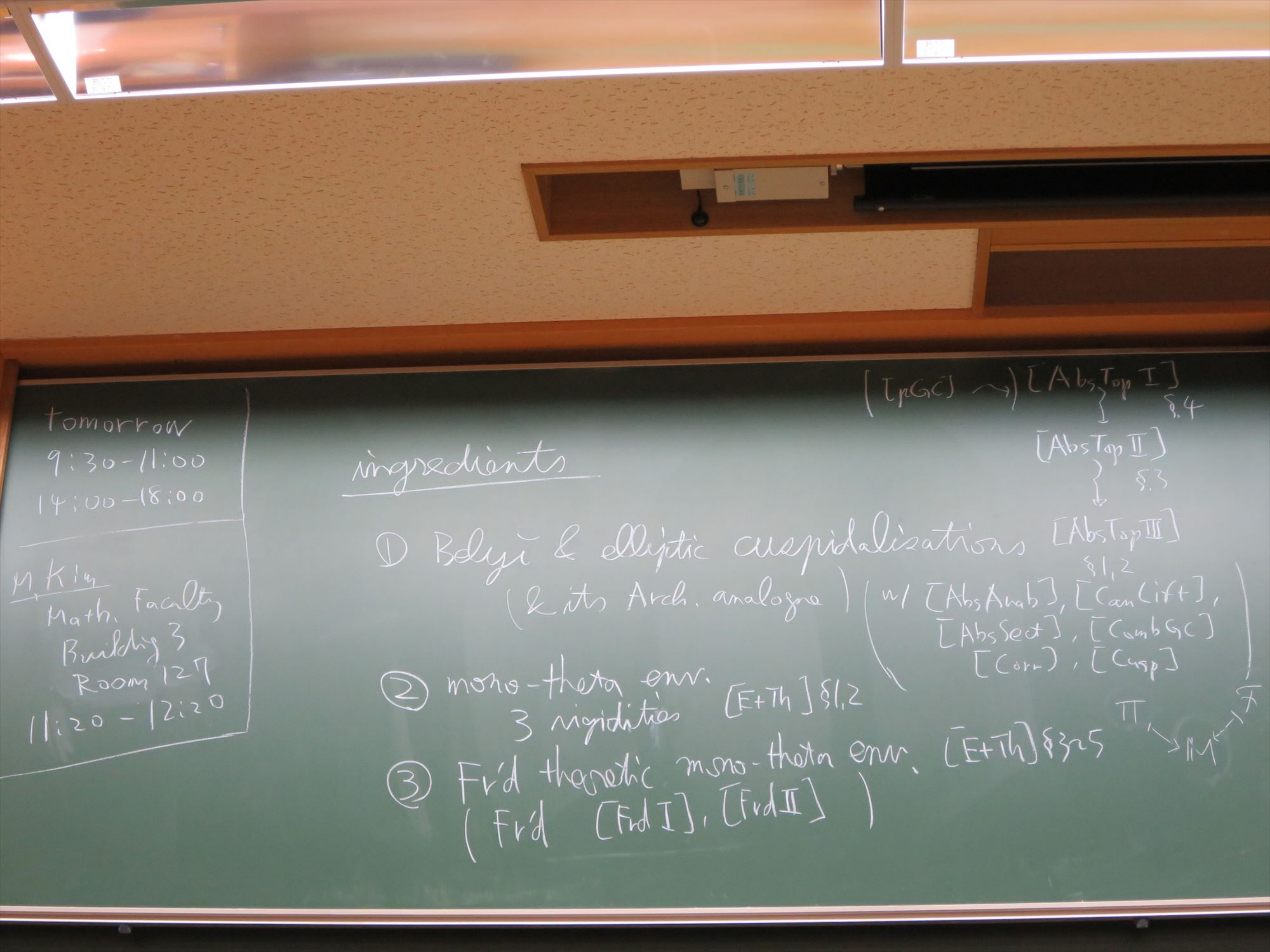
(← deep & need delicate constructions)

calculate concretely ([IUTchIV])

$0 \lesssim -(ht) + (\text{indet.})$

$\parallel$ log-diff (+ log.-cond.)

$\rightsquigarrow$ $ht \lesssim$ log-diff. + log.-cond. Vojta's inequality


tomorrow
9:30-11:00
14:00-18:00
M. Kim
Math. Faculty
Building 3
Room 127
11:20-12:20
ingredients
① Belyi & elliptic cuspidalisations
(& its Arch. analogue)
w/ [AbsAnab], [CanLift],
[AbsSect], [CombGC]
[Corr], [Cusp]
([pGC] → [AbsTop I]
§4
[AbsTop II]
§3
[AbsTop III]
§1,2
② mono-theta env.
3 rigidities [E+Th] §1,2
③ Fr'd theoretic mono-theta env. [E+Th] §3~5
(Fr'd [Frd I], [Frd II])
Π
M
X

③ Fr'd theoretic mono-theta env. [E+Th] §3,5
(Fr'd [FrdI], [FrdII])

④ semi-graphs of anabelioids [SemiAnab]
(~ [Anab])

⑤ log-shells [AbsTopIII] §3,4

⑥ reductions by usual arith. geom.
[GenEll] ~ [Belyi]

mol. conj.
vs temp. conj.

loc. temp
△ ~ temp.

gl.
△
mod.

$\int e^{-x^2} dx$

cf. [Pano]
[HASurI], [HASurII]

§2. Preliminaries on Anabelian Geom.

§2.1 Some Basics on Galois Group of local fields

Prop 2.1 $i = 1, 2$, $K_i / \mathbb{Q}_{p_i}$ fin. w/ res. field k_i

$\overline{K_i}$: alg. closure of K_i w/ res. field $\overline{k}_i$

$e(K_i)$: ram. index of K_i over $\mathbb{Q}_{p_i}$

$f(K_i) := [k_i : \mathbb{F}_{p_i}]$, Put $G_{K_i} := \mathrm{Gal}(\overline{K}_i / K_i)$

$(G_{K_i} \supset) I_{K_i}$: inertia $\supset P_{K_i}$: wild inertia

$\alpha : G_{K_1} \xrightarrow{\sim} G_{K_2}$ isom. of prof. gps

$\alpha : G_{K_1} \xrightarrow{\sim} G_{K_2}$ isom. of prof. gps

(1). $p_1 = p_2$ $(=: p)$

(2). $\alpha^{ab} : G_{K_1}^{ab} \xrightarrow{\sim} G_{K_2}^{ab}$ & $k_1^\times \subset O_{K_1}^\times \subset K_1^\times \overset{\text{LCFT}}{\subset} G_{K_1}^{ab}$

induce isom's

(a). $\alpha^{ab} : k_1^\times \xrightarrow{\sim} k_2^\times$

(b). $\alpha^{ab} : O_{K_1}^\times \xrightarrow{\sim} O_{K_2}^\times$

(c). $\alpha^{ab} : O_{K_1}^\triangleright \xrightarrow{\sim} O_{K_2}^\triangleright$

(d). $\alpha^{ab} : K_1^\times \longrightarrow K_2^\times$

(3) (a) $[K_1 : \mathbb{Q}_p] = [K_2 : \mathbb{Q}_p]$

(b) $f(K_1) = f(K_2)$

(c) $e(K_1) = e(K_2)$

(4) the restrictions of α induce

(a) $\alpha|_{I_{K_1}} : I_{K_1} \xrightarrow{\sim} I_{K_2}$

(b) $\alpha|_{P_{K_1}} : P_{K_1} \xrightarrow{\sim} P_{K_2}$

(5) the induced map $G_{K_1}^{ab} / \mathrm{Im}(I_{K_1}) \xrightarrow{\sim} G_{K_2}^{ab} / \mathrm{Im}(I_{K_2})$ preserves the Frob. elt Frob_{K_i}

$\mathbb{P}_{K_1}: I_{K_1} \to \mathbb{P}_{K_2}$

(5) the induced map $G^{ab}_{K_1}/\mathrm{Im}(I_{K_1}) \xrightarrow{\sim} G^{ab}_{K_2}/\mathrm{Im}(I_{K_2})$ preserves the Frob. elt Frob_{K_i}

(6). the collection of the isom's $\{(\alpha|_{U_1})^{ab}: U_1^{ab} \xrightarrow{\sim} U_2^{ab}\}_{U_1 \xrightarrow{\sim} U_2}$ induces $\mu_{\mathbb{Q}/\mathbb{Z}}(\overline{K}_1) \xrightarrow{\sim} \mu_{\mathbb{Q}/\mathbb{Z}}(\overline{K}_2)$

$$\begin{array}{ccc} U_1 & \xrightarrow{\sim} & U_2 \\ \cap_{\text{open}} & & \cap_{\text{open}} \\ G_{K_1} & \xrightarrow[\alpha]{\sim} & G_{K_2} \end{array}$$

which is compat. w/ the actions of G_{K_i} $(i=1,2)$ via $\alpha: G_{K_1} \xrightarrow{\sim} G_{K_2}$

In particular, α preserves the cycl. char's $\chi_{cyc,i}$ $(i=1,2)$

$k = \overline{k}$ char $= 0$

$\mu_{\widehat{\mathbb{Z}}}(k) := \varprojlim_n \mu_n(k)$ ← gp of n-th roots of k

$\mu_{\mathbb{Q}/\mathbb{Z}}(k) := \mu_{\widehat{\mathbb{Z}}}(k) \otimes_{\widehat{\mathbb{Z}}} \mathbb{Q}/\mathbb{Z}$

(7) the isom $\alpha^*: H^2(\mathrm{Gal}(\overline{K}_2/K_2), \mu_{\mathbb{Q}/\mathbb{Z}}(\overline{K}_2)) \xrightarrow{\sim} H^2(\mathrm{Gal}(\overline{K}_1/K_1), \mu_{\mathbb{Q}/\mathbb{Z}}(\overline{K}_1))$ induced by α is compat. w/ the isom's $H^2(\mathrm{Gal}(\overline{K}_i/K_i), \mu_{\mathbb{Q}/\mathbb{Z}}(\overline{K}_i)) \xrightarrow{\sim} \mathbb{Q}/\mathbb{Z}$ $(i=1,2)$

proof) Hoshi's talk.

↑

Rem constructions are made in mono-anabelian manner

"group theoretic" algorith

$$G_1 \simeq G_2$$

$$\rightsquigarrow$$

$$F(G_1) \xrightarrow{\sim} F(G_2)$$

$$G \rightsquigarrow F(G)$$

$$\text{\rotatebox{90}{$\simeq$}}$$

$$Gal(\bar{F}/F)$$

§2.2 Arithmetic Quotients

Prop 2.2 ([AbsAnab, Lem 1.1.4])

F : field, $\overline{F} \supset F$ sep. closure

$G := \mathrm{Gal}(\overline{F}/F)$

$$1 \to \Delta \to \Pi \to G \to 1$$

exact seq. of prof. gps

Assume Δ : top. fin. gen.

(1). F : NF

Then Δ is gp-th'ally characterized in Π by the max. closed normal subgp of Π which is top. fin. gen.

proof) Hoshi's talk.

$\mathrm{Hom}(-, \mathbb{Q}/\mathbb{Z})$ $\quad 0 \to (\Delta^{ab})_G \to \Pi^{ab} \to G^{ab} \to 0$

$G \supset G'$ open $\leadsto F'/F$

(2). $F/\mathbb{Q}_p$ fin.

For open subgp $\Pi' \subset \Pi$, $\Delta' := \Pi' \cap \Delta$, $G' := \Pi'/\Delta'$

let G' act on $(\Delta')^{ab}$ by conj.

Assume (Tam1) $\forall \Pi' \subset \Pi$ open, $Q := ((\Delta')^{ab})_{G'}/(\mathrm{tor})$ is a fin. gen. free $\hat{\mathbb{Z}}$-module

($(\Delta')^{ab})_{G'}$: coinv.)

Then Δ is gp th'cally characterized in Π as the intersection of those open subgps $\Pi' \subset \Pi$ s.t. for $\forall$ l, $l \neq p$

(Tam2) $\dim_{\mathbb{Q}_p} (\Pi')^{ab} \otimes_{\hat{\mathbb{Z}}} \mathbb{Q}_p - \dim_{\mathbb{Q}_l} (\Pi')^{ab} \otimes_{\hat{\mathbb{Z}}} \mathbb{Q}_l$

$$= [\Pi : \Pi'] (\dim_{\mathbb{Q}_p} (\Pi)^{ab} \otimes_{\hat{\mathbb{Z}}} \mathbb{Q}_p - \dim_{\mathbb{Q}_l} (\Pi)^{ab} \otimes_{\hat{\mathbb{Z}}} \mathbb{Q}_l).$$

where p is also gp-th'ically characterized as the unique prime s.t.

$$\dim_{\mathbb{Q}_p}(\Pi)^{ab} \otimes_{\hat{\mathbb{Z}}} \mathbb{Q}_p - \dim_{\mathbb{Q}_\ell}(\Pi)^{ab} \otimes_{\hat{\mathbb{Z}}} \mathbb{Q}_\ell \neq 0$$

for $\exists \infty \ell$.

proof) (1). $Gal(\bar{F}/F) \supset$ top. fin. gen. closed normal subgp $= \{1\}$

Fact cf. [FJ, Th 15.10]

(2). $1 \to \Delta \to \Pi \to G \to 1$

$\rightsquigarrow 1 \to H^1(G, \mathbb{Q}/\mathbb{Z}) \to H^1(\Pi, \mathbb{Q}/\mathbb{Z}) \to H^1(\Delta, \mathbb{Q}/\mathbb{Z})^G \to H^2(G, \mathbb{Q}/\mathbb{Z})$

$\mathrm{Hom}(-, \mathbb{Q}/\mathbb{Z})$: $0 \to (\Delta^{ab})_G \to \Pi^{ab} \to G^{ab} \to 0$ $\quad$ ($H^2(G,\mathbb{Q}/\mathbb{Z}) = 0$)

$G \supset G'$ (open) $\leftrightarrow F'/F$

(2). $F/\mathbb{Q}_p$ fin.

For open subgp $\Pi' \subset \Pi$, $\Delta' := \Pi' \cap \Delta$, $G' := \Pi'/\Delta'$

let G' act on $(\Delta')^{ab}$ by conj.

(Tam 1) $\rightsquigarrow$ $\dim_{\mathbb{F}_p}(\Pi')^{ab}\otimes_{\mathbb{Z}}\mathbb{F}_p - \dim_{\mathbb{F}_\ell}(\Pi')^{ab}\otimes_{\mathbb{Z}}\mathbb{F}_\ell$

$$= \dim_{\mathbb{F}_p}(G')^{ab}\otimes_{\mathbb{Z}}\mathbb{F}_p - \dim_{\mathbb{F}_\ell}(G')^{ab}\otimes_{\mathbb{Z}}\mathbb{F}_\ell \underset{\text{LCFT}}{\overset{\ell \neq p}{=}} [F':\mathbb{Q}_p]$$

$\rightsquigarrow$ gp thic char. of p

(Tam 2) $\Leftarrow$ $[F':\mathbb{Q}_p] = [\Pi:\Pi'][F:\mathbb{Q}_p]$

$\Leftarrow$ $[\Pi:\Pi'] = [G:G']$

$\Leftarrow$ $\Delta = \Delta'$

//

Lem 2.3 ([AbsAnab, Lem 1.1.5])

$F/\mathbb{Q}$ fin., A: semi-abelian var. $/F$

$\overline{F} \supset F$ alg. closure $G := \mathrm{Gal}(\overline{F}/F)$

$T(A) := \mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, A(\overline{F}))$ Tate module of A

$\Longrightarrow Q := T(A)_G/(\mathrm{tor})$ is a fin. gen. free $\hat{\mathbb{Z}}$-module.

$1 \to \Delta \to \Pi \to G \to 1$

exact seq. of prof. gps

Assume Δ: top. fin. gen.

(1). F: NF

Then Δ is gp-th'ally characterized in Π by the max. closed normal subgp of Π which is top. fin. gen.

Cor X: geom. conn. smooth hyperb. curve / F /$_{\mathbb{Q}}^{\text{fin.}}$

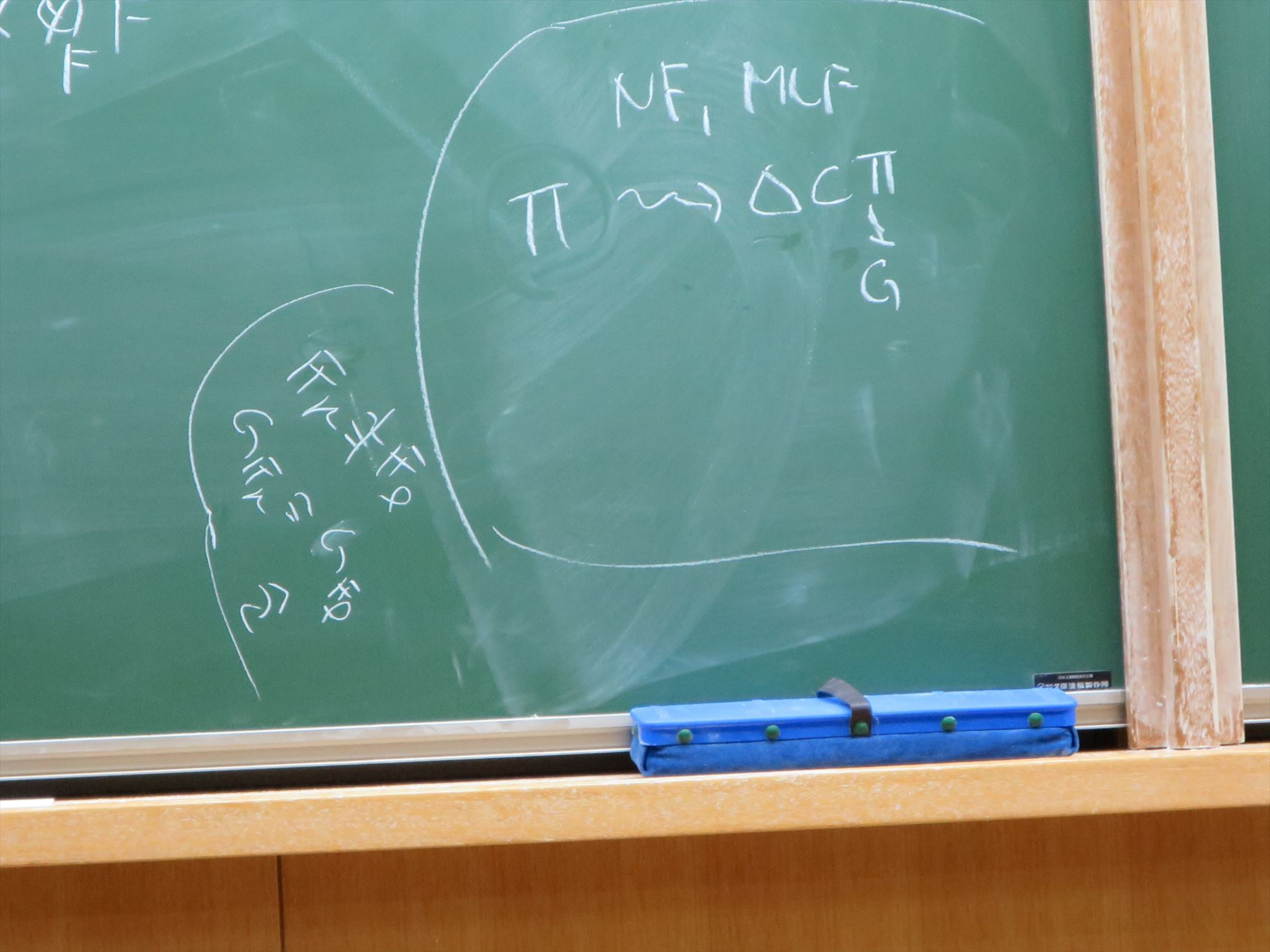
$\exists$ We have a gp th'c characterization of $\Delta := \pi_1(X_{\overline{F}}, \overline{x})$ in $\Pi := \pi_1(X, \overline{x})$

$X_{\overline{F}} = X \otimes_F \overline{F}$

NF, MLF

$\Pi \rightsquigarrow \Delta \subset \Pi$

$\Pi \twoheadrightarrow G$


NF, MCF
G